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Remarks on Mesh Quality

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Various aspects of mesh quality are surveyed to clarify the disconnect between the traditional uses of mesh quality metrics within industry and the fact that quality ultimately depends on the solution to the physical problem. Truncation error analysis for finite difference methods reveals no clear connection to most traditional mesh quality metrics. Finite element bounds to the interpolation error can be shown, in some cases, to be related to known quality metrics such as the condition number. On the other hand, the use of quality metrics that do not take solution characteristics into account can be valid in certain circumstances, primarily as a means of automatically detecting defective meshes. The use of such metrics when applied to simulations for which quality is highly-dependent on the physical solution is clearly inappropriate. Various flaws and problems with existing quality metrics are mentioned, along with a discussion on the use of threshold values. In closing, the author advocates the investigation of explicitly-referenced quality metrics as a potential means of bridging the gap between a priori quality metrics and solution-dependent metrics.

1. Introduction

This paper is based on a short course on 'mesh quality' that has been presented by the author over the years, primarily at the International Meshing Roundtable. The short course gives an overview on what mesh quality is, how it is measured, and how the topic relates to broader topics such as mesh generation, mesh improvement, and solution-adaptivity. The course has roots primarily within what is called a priori or non-adaptive mesh generation as it evolved within the Finite Element community. Finite element mesh generation codes such as Cubit, Fluent, SDRC/IDEAS, and PATRAN, are commonly used to create initial meshes on complicated geometries arising from assemblies of mechanical parts such as those found in automobiles, nuclear reactors, and even particle accelerators. In the automobile industry, for example, a common application of finite elements and meshes is that of crash simulations involving non-linear computational mechanics. Over the years, automotive engineers have devised certain mesh quality metrics (e.g., Robinson[21] and Field[26]) which are used to automatically screen out defective meshes that are a priori unsuitable for a crash simulation. The metrics measure such things as whether or not an individual element within a mesh is inverted (or folded/tangled), has excessively small or large angles, poor shape, and other undesirable properties. If a defective mesh is generated, the mesh is either discarded (a new one is generated by modifying the meshing procedure) or improved via local improvement techniques such as smoothing or swapping/ flipping of elements. The use of these metrics has spread beyond their initial intent into different realms of engineering (e.g., CFD and heat transfer), into alternative discretization methods such as finite volume and even finite differences, and, to some extent, into inappropriate settings.

Although the relationship between mesh quality assessment and initial mesh generation is fairly straightforward, the relationship of mesh quality assessment to solution-adaptivity is not. Mesh quality metrics tend to be formulated on the basis of some geometric criterion (shape, size, angles, aspect ratio) that is independent of the solution to the physical problem. As such, these mesh quality metrics appear quite irrelevant to the problem of creating a mesh that results in acceptable discretization error on a particular simulation. In the short course, this disconnect between a priori quality metrics and solution-adaptivity is recognized, so

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material from truncation error analyses and the finite element theory for interpolation error is included to provide insight into how meshes impact solution accuracy. In addition, research into mesh quality metrics which can be referenced to solution characteristics has been initiated.

Brief remarks on a provisional definition of mesh quality are made in Section Two. In section Three a review of basic results on truncation error within the finite difference method is presented, which draws on results derived over twenty years ago by Mastin and others. In section Four, basic results on interpolation error within the finite element theory are reviewed, partly to contrast with the results in finite differences, but more importantly, to convey the message that mesh quality ultimately depends on the relationship between the mesh and the physical solution. Section Five reviews the traditional uses of mesh quality metrics, points out some short-comings of common metrics, and distinguishes between valid and invalid uses of such metrics. Section Six considers explicitly-referenced quality metrics that can incorporate information about the physical solution which may be of value in both a priori and in solution-adaptive meshing.

2. What is Mesh Quality?

Make no mistake about it, mesh quality can have a large influence upon the accuracy (and efficiency) of a simulations based on the solution of partial differential equations (PDE)'s. Many factors go into the influence of mesh on accuracy including the type of physics being simulated, details of the solution to the particular simulation, the method of discretization, and geometric mesh properties having to do with spacing, curvature, angles, smoothness, etc. But what, exactly, is mesh quality? Some would say it is a little bit like pornography in that one knows it when one sees it, but can't define it ahead of time. As far as the author is aware, there are no formal definitions of mesh quality appearing in the literature so, for the purposes of this paper, the following provisional definition is adopted.[a]

Mesh Quality concerns the characteristics of a mesh that permit a particular numerical PDE simulation to be efficiently performed, with fidelity to the underlying physics, and with the accuracy required for the problem.

This definition hints at several issues. First, mesh quality depends on the particular calculation which is undertaken and thus changes if a different calculation is performed. Second, a mesh should do no harm, i.e., it should not create difficulties for the simulation. For example, inverted elements can cause a loss of fidelity or even cause the simulation to halt prematurely. The mesh should not contribute to ill-conditioning of the matrix system solved by the application, nor should it result in clusters of large eigenvalues which can cause slow convergence rates. The mesh should not cause the numerical solution to exhibit mesh imprinting (of course, the discretization method could be the ultimate culprit in this). Third, the mesh should result in sufficiently accurate simulations, i.e., those which are in the asymptotic regime, and those which reduce both global and local error below the required level. Ultimately, the mesh and discretization method together must enable the simulation to satisfy the requirement that the size of the error bars due to problem discretization are acceptable.

Given this definition as a general guideline, let us take a look at how mesh quality manifests itself in various settings such as error analyses in finite differences and finite elements, a priori mesh generation, and mesh quality improvement.

3. Mesh Quality in Finite Differences

This section is based on a body of literature that appeared in the 80's and early 90's on the relationship between mesh and truncation error in finite differences (see for example, Mastin,[1] Thompson & Mastin,[2] Lee & Tsuei,[3] and Huang & Prosperetti[4]). These papers assume that the mesh is derived from a curvilinear coordinate system based on a map between a logical and the physical domain. The result is a block-structured mesh. If the logical coordinates are denoted by ξ_j, with $1 \leq j \leq J$, and the physical coordinates by $x_i = x_i(\xi_j)$, with $1 \leq i \leq I$, then the Jacobian (matrix) of the map is $J_{ij} = \partial x_i / \xi_j$. This matrix plays a

[a]The definition excludes non-PDE applications of meshes such as their use in visualization because in the latter case, the requirements on quality are much simpler.

critical role in finite difference mesh quality because it can be used to measure local tangent lengths, angles, volume, and orientation. If one transforms the Laplace operator $\nabla_x \phi = \nabla_x \cdot \nabla_x \phi = \partial^2\phi/\partial x_i^2$ on the physical domain to the logical domain, one can express the operator as $\frac{1}{det(J)}\frac{\partial}{\partial \xi_i}(J^{-t})_{ij}\frac{\partial \phi}{\partial \xi_j}$ (with summation over the indices). The Laplace operator is, of course, elliptic and it is thus desired that the transformed operator also be elliptic. This property is guaranteed provided $det(J) > 0$ over the domain. In terms of mesh, the requirement becomes the requirement of invertability. In general, meshes with locally negative volume (Jacobian determinant) can cause non-physical behavior due to a change in type of the equation solved.

In finite differences, one relies on Taylor Series expansions of the dependent variables in the physical equation to derive a discretization. For a uniform mesh in one-dimension, the series is truncated after a few terms, resulting in a discretization error of the form Ch^p, plus higher order terms. Here, p is the order of the error and C is a constant that is independent of the local mesh size, $h = \Delta x$. The accuracy of the PDE solution thus locally depends on the mesh size parameter h; in general, the smaller the mesh size, the more accurate the calculation. Solution-adaptive h-refinement, of course, exploits this fact. In general, an adaptive mesh method based on equidistribution of the error would require h to be relatively small in portions of the mesh where C is large, and relatively large h where C is small. Of course, the situation is much more complicated in multiple dimensions, when there are multiple dependent variables to which one needs to adapt, and when high-order accuracy is demanded.

The mesh also impacts accuracy through the constant C because the latter depends both on derivatives of the dependent variables and on derivatives of the mesh. For example, for a non-uniform mesh in one-dimension, the centered difference approximation to $d\phi/dx = (d\xi/dx)(d\phi/d\xi)$ has a leading truncation error term of the form

$$\frac{(\Delta\xi)^2}{6}\frac{\phi_{\xi\xi\xi}}{x_\xi} = \frac{(\Delta\xi)^2}{6}\left(x_\xi^2\phi_{xxx} + 3x_{\xi\xi}\phi_{xx} + \frac{x_{\xi\xi\xi}}{x_\xi}\phi_x\right)$$

C in this example depends on the first, second, and third derivatives of both the mapping and the function; therefore mesh quality depends in a complicated way on discretizations of these quantities. There is no direct geometric interpretation of the error in terms of a single a priori mesh quality metric such as stretching or volume. In higher dimensions, the connection of the truncation error to simple mesh metrics such as skew, angle, aspect ratio, volume, etc. is even more obscure. To control the leading term in the truncation error of the first derivative clearly requires proper coordination of the mesh with the solution derivatives.[b]

4. Mesh Quality in Finite Elements

This section is based on portions of the finite element literature related to interpolation error (see for example, Ciarlet,[5] Shewchuk,[6] Nadler,[7] D'Azevedo & Simpson,[8] Formaggia & Perotto,[9] Jamet,[10] Ciarlet & Raviart,[11] Fried,[12] Du et. al.,[13] Batdorf et. al.,[14] and Xu & Zikatanov[15]). Interpolation error (and corresponding bounds) in finite elements are defined at the element level and thus can be connected to element-based mesh quality metrics. However, for the most part, element metrics arising from the interpolation error bounds do not correspond to the commonly-used solution-independent quality measures, in part because the metrics derived from the bounds can also factor in the effect of the physical solution.

The traditional emphasis in finite element interpolation theory is placed on the asymptotic behavior of bounds on the error, so that convergence of the finite element method can be proved. Quality metrics that appear in the bounds are thus relevant primarily in the asymptotic regime; an exception would be the tight bounds of Shewchuk[6] which hold even in the non-asymptotic case. The basic theory for interpolation error on *simplicial* mesh elements begins with the master and physical mesh elements, denoted by K' and K, respectively. Assume that there exists an invertible affine mapping $F(x') = A_K x' + a_k$ from $x' \in K'$ to $x \in K$. A_K is a $d \times d$ matrix (with $d = 1, 2$, or 3 usually) that is constant over the element and contains mesh-related information such as size, shape, and orientation. Let v be the dependent variable in a PDE and

[b]A similar situation likely holds with finite volume discretizations, but the author presently has not taken the time to investigate.

assume it belongs to the Sobolev space $W^{m,p}$. A semi-norm can be defined using the multi-index notation

$$|v|^p_{m,p,K} = \sum_{|\alpha|=m} \int_K \left| \frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}} \right|^p dx$$

The local interpolation error on the element is expressed in terms of the semi-norm, namely

$$\varepsilon_{m,p,K} = |v - \Pi v|_{m,p,K}$$

where Π is the interpolation operator. Ciarlet derived the following bound on the interpolation error:

$$|v - \Pi v|_{m,p,K} \le C |A_K|_2^r |A_K^{-1}|_2^m |det(A_K)|^{\frac{1}{p}-\frac{1}{q}} |v|_{r,q,K}$$

with C a constant that is independent of K, and $0 \le m \le r \le k+1$, $p \ge 1$, $q \ge 1$, with k the degree of the approximating polynomial. For linear interpolation, $k = 1$. In that case, a useful choice for the parameters r and q is $r = 2$ and $q = p$, and the bound reduces to

$$|v - \Pi v|_{m,p,K} \le C \kappa_2^m(A_K) |A_K|_2^{2-m} |v|_{2,p,K}$$

where $\kappa_2(A) = |A|_2 |A^{-1}|_2$ is the *condition number* of the matrix. Finally, the error in the solution function itself corresponds to $m = 0$, while the gradient of the error in the function corresponds to $m = 1$, thus giving the bounds

$$\begin{aligned} |v - \Pi v|_{0,p,K} &\le C |A_K|_2^2 |v|_{2,p,K} \\ |v - \Pi v|_{1,p,K} &\le C \kappa_2(A_K) |A_K| |v|_{2,p,K} \end{aligned}$$

One sees from the finite element interpolation error bounds that certain convenient element-based mesh quality metrics such as $|A_K|^2$ and condition number appear naturally as part of the bounds.

An important observation on the first bound immediately above is that the interpolation error on the function v goes like $|A_K|^2$, which is equivalent to saying the error goes like the square of the circumradius of the element. Thus, to decrease the function error, one must refine or create smaller elements. For the second bound, dealing with the gradient of the interpolation error, the error goes like the circumradius times the condition number of the element. As the element becomes degenerate due to some internal angle approaching π, the latter bound goes to infinity. Thus to bound the gradient of the interpolation error requires one to maintain good element shape, in addition to size.

Note that the condition number and the $|A_K|^2$ 'metric' derived from the bounds are solution-independent metrics and can thus be applied in an a priori fashion. In contrast, one can consider tighter bounds derived by Formaggia & Perotto[9] in which the metrics that derive from the bounds are solution-dependent

$$\begin{aligned} |v - \Pi v|_{0,2,K} &\le C |A_K^t D^2(v) A_K|_2 \\ |v - \Pi v|_{1,2,K} &\le C |A_K^{-1}| |A_K^t D^2(v) A_K|_2 \end{aligned}$$

where $D^2(v)$ is the Hessian matrix of the physical variable. The expressions to the right of the constant C in these bounds can be considered to be solution-dependent metrics. Section 6 discusses other metrics that are not directly derived from interpolation error bounds, but which can be used as solution-dependent metrics.

There are, of course, many additional error bounds presented within the finite element theory. The bounds of Shewchuk[6] strongly suggest certain quality metrics. Other bounds suggest metrics that are equivalent to the ones presented here (e.g., Jamet & Acosta[10]). Practicing engineers are often more interested in the local maximum errors $\varepsilon_{0,\infty,K}$ and $\varepsilon_{1,\infty,K}$ in the function and gradient. Rajan[16] showed that, in any dimension, the Delaunay triangulation is optimal for minimizing $\varepsilon_{0,\infty,K}$. The finite element bounds for quadrilateral, hexahedral, and curved elements are less well-developed.

A few remarks on solution-adaptive meshing algorithms within the context of finite elements are in order. Many of these algorithms rely on h-refinement to reduce error. The effect of the constant C is neglected, often (but not always) appropriately. For a given required level of accuracy, h-refinement is an economical approach to reducing the necessary number of degrees of freedom. Mesh quality metrics have little to contribute to this goal since they tend to measure properties embedded in the constant C, rather than mesh size. As a consequence, the metrics cannot tell us where in the domain the mesh should be refined (that role is reserved for error estimators or indicators).

Another way to look at adaptive meshing is to ask: for a fixed number of degrees of freedom, what is the mesh that provides the most accurate calculation? The upper bounds to the interpolation error then become relevant because, by assembling local bounds into a global measure of error, they can provide guidance on which of several meshes containing similar degrees of freedom gives the least error. From this perspective, mesh quality metrics such as those appearing in the Ciarlet bounds are relevant.

5. Traditional Uses of Mesh Quality Metrics

As described in the Introduction, there are a large number of a priori mesh quality metrics that are used primarily to screen out defective meshes before one begins an analysis or simulation. Such metrics can be found, for example, in the PATRAN reference manual, the SDRC/IDEAS users guide, and the Fluent FIMESH user's manual. Additional references include Robinson,[21] Parthasararthy,[22] Oddy,[23] Pebay,[24] Pebay & Baker,[25] Field,[26] and Knupp.[29] A compendium of such metrics has been implemented within the VERDICT software library.[27] Most of these metrics are element-based, meaning that they define the quality of a finite element or perhaps a cell within a finite volume mesh. Similar metrics can be derived for the case of curvilinear coordinates, wherein the metrics depend primarily on combinations of the first and second derivatives of the mapping. Use of the mapping-based metrics appears to be less widespread, even when curvilinear coordinates are involved.

In nearly all instances, a mesh quality metric can be viewed as a scalar function of local element or vertex positions within a mesh. The commonly used metrics measure some geometric or other local property that is independent of the solution to the physical problem. The phrase 'mesh quality metric' is somewhat of a misnomer since more properly the phrase should be 'element quality metrics' or 'local quality metrics'. To measure the quality of the global mesh itself, one usually resorts to a discrete norm approach say, for example, the ℓ_2 norm of the vector consisting of all the local mesh qualities.

Mesh quality metrics are convenient because they can be automatically computed by looping over the elements within a mesh, thus removing the tedium of performing a visual inspection of a small mesh and rendering practical the assessment of the quality of large 3D meshes. What are the valid uses of such metrics? Four are described:

- *Automatic defect detection.* Application of the metrics to an initially generated mesh permits identification of mesh defects such as non-invertible elements, large or small angles, and bad mesh topology. Such meshes are screened out and never used in the simulation.

- *Mesh Generation Results Assessment Based on Engineering Judgment.* For simulations that are performed over and over, with minor variations between calculations, engineers tend to develop expertise as to what constitutes a good mesh for the problem at hand. Some of this expertise is manifested as 'rules of thumb' that translate to requirements on mesh quality. Quality metrics can help ensure these rules are obeyed by the mesh used in the simulation.

- *Non-adaptive Calculations.* Adaptive mesh generation is non-essential if the physical solution is known to be nearly isotropic and to vary slowly over the domain. In this case one can assess mesh quality independent of the solution because the ideal mesh element is the same over the whole mesh.

- *A Priori Mesh Improvement Methods.* If inadequate meshes are identified by applying mesh quality metrics in a valid manner, one may wish to improve the quality via local edge or element swapping, or by node-movement strategies such as mesh smoothing or optimization. Mesh quality metrics are useful in guiding such improvement techniques.

The primary *invalid* use of mesh quality metrics is to use them to evaluate a mesh independently of solution-knowledge when the solution is either anisotropic, not slowly-varying over the domain, or both. For example, it is incorrect to assess the quality of a mesh used in a boundary layer CFD calculation based on the criterion that all elements be perfect squares or cubes.

Another common pitfall in using mesh quality metrics concerns the threshold criteria for determining if quality is sufficient. For example, suppose the scalar range of a given quality metric runs from zero to one, with one being a perfect element and zero being a very bad element in terms of quality. The usual procedure is to reject any element whose quality metric value is less than some threshold T, with $0 \leq T < 1$. The crucial question is, what is the value of T, and what is the justification for choosing that value? As far as the author knows, the value of T is never justified on the basis of any mathematical analysis, but instead is selected rather arbitrarily, in some cases pulled out of thin air. To some extent, this lack of justification of the threshold value makes the use of quality metrics arbitrary.

There are other problems associated with some mesh quality metrics that have been proposed in the past. These include:

- definitions which change depending on whether one is evaluating a two- or three-dimensional element. For example, it is clear how to measure angles of two-dimensional elements, but in 3D one can measure either face angles or dihedral angles. Which is more important?
- definitions which change depending on whether one is evaluating the quality of a simplicial element or a non-simplicial element. For example, one measure of the quality of a quadrilateral is to compute the ratio of the lengths of the two diagonals. This has no clear analogy on triangles.
- definitions which become meaningless as an element changes shape. For example, the meaning of aspect ratio is clear when applied to a rectangular element, but makes little sense on quadrilaterals, especially if one or more of the angles in the element is small and the others large.
- definitions which result in metrics whose ideal value does not necessarily imply the ideal element. For example, the quadrilateral aspect ratio metric of Robinson is

  $$AR = \max\left(\frac{|e_x|}{|e_y|}, \frac{|e_y|}{|e_x|}\right)$$

  with

  $$\begin{aligned} e_x &= \frac{1}{2}[(x_1 - x_0) + (x_2 - x_3)] \\ e_y &= \frac{1}{2}[(x_3 - x_0) + (x_2 - x_1)] \end{aligned}$$

  The range of the metric is 1.0 to ∞, with $AR = 1$ if the element is a square *or if the element is a rhombus (kite)*! This is not the only metric which suffers from this kind of flaw.
- definitions which assume a certain element configuration is always ideal. For example, consider the aspect ratio metric on a rectangular element with with w and height h. The usual definition of aspect ratio would be $AR = h/w$, which ranges from zero to infinity. The ideal value of the metric is often assumed to be 1.0, indicating a square is the ideal rectangle. Clearly, for many calculations this is not the right ideal rectangle to use. One can redefine the aspect ratio to account for this by setting $AR = h/(sw)$, with the value of s indicating the aspect ratio of an arbitrary ideal rectangle. Thus, if $s = 10$, the metric value is 1.0 when the rectangle has the ratio $h/w = 10$. The second definition of aspect ratio here is termed an *explicitly* referenced metric, whereas the first definition is implicitly referenced because it is assumed that 1.0 is always the ideal. Explicitly referenced metrics are rare in traditional mesh quality metric definitions, however this idea has been recently developed by the author (see next section).

In summary, the current status of a priori mesh quality metrics is that there are a large number of redundant metrics with some (but not all) of them having flawed definitions. Most have implicit references, lack justifiable threshold values, and have no connection to the solution to the physical problem. Thus, their valid uses are confined to the detection of defective meshes, to problems where expert judgment is viable, and in some applications of mesh quality improvement methods. More recent investigations into mesh quality have developed the idea of explicitly referenced quality metrics which hold the promise of being able to tie metrics more closely to solution properties.

6. Explicitly-Referenced Algebraic Mesh Quality Metrics

Explicitly-referenced quality metrics essentially contain a user-defined or application-specific reference which translates local geometric quality to quality characteristics that are relevant to specific application problems. In the theory of algebraic mesh quality metrics,[20] this is accomplished by using *target*-matrices W as the explicit reference.[c] Target matrices W are essentially a transformation from a reference element K' to an ideal (or desired) element K''. Euclidean properties of the physical element are described by the so-called *active*-matrix A which gives the transformation from the reference element to the physical element K. Figure 1 illustrates the situation in two dimensions. The weighted active-matrix, $T = AW^{-1}$, transforms the ideal element to the physical element. The use of the symbol A to denote the active-matrix is not accidental because it is precisely the same A_K that appears in the finite element theory presented in Section 4. The goal then is to create local mesh quality metrics that take a given matrix T to a scalar that measures local quality with respect to the ideal element, thus creating explicitly referenced metrics.[d]

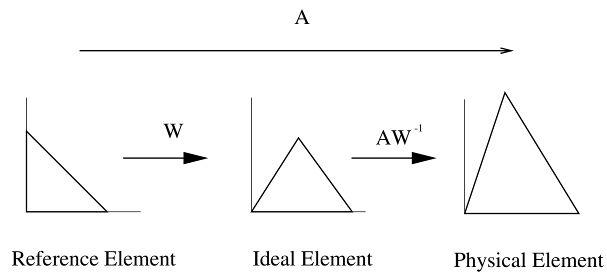

Figure 1. Relation between active, target, and weighted active matrices in the target-matrix paradigm

An important example of an explicitly-referenced algebraic mesh quality metric is the *condition number* shape metric μ:

$$\mu(T) = \frac{d}{\kappa(T)}$$

with $d = 2$ or 3 being the dimension of the element and $\kappa(T) = |T||T^{-1}|$. Usually the matrix norm is taken to be the Frobenius norm, but equivalent norms can also be used. The metric measures the distance of T to the set of singular matrices, with $\mu = 0$ denoting degeneracy and $\mu = 1$ denoting that the physical element is the same as the ideal, up to an arbitrary scaling and rotation. Note that the condition number metric, which appears in the Ciarlet finite element bound on the gradient interpolation error, is scale and orientation invariant and thus constitutes what is known as a *shape* metric (see Dompierre[28]). Shape metrics essentially measure a combination of angle and aspect ratio. Using T in the argument of the condition number, instead of A, converts the implicitly referenced metric in the Ciarlet bound to an explicitly-referenced metric. Thus $\kappa(T)$ does not assume that the ideal element is always an equilateral triangle or tetrahedron, unlike most of the other shape metrics in the literature.

Other explicitly referenced metrics are described in Knupp.[31] As a example, the metric $\mu(T) = |T - I|^2$ can be used in updating a mesh on a deforming domain (Knupp[32]), while the metric $\mu(A, W) = |A^{-1} - W^{-1}|^2$ was used to align the mesh with a vector field (Knupp[33]). In the deforming domain case, the target-matrix is

[c] The target and other matrices are small, being either 2×2 or 3×3 matrices corresponding to whether an element is two- or three-dimensional.

[d] This idea can be carried over to non-simplicial elements by using more than one sample point location within an element (see Knupp[30]).

obtained from the mesh on the undeformed domain, thus the metric measures the deviation of the updated mesh from the initial undeformed mesh. In the alignment case, the target-matrices were obtained from the solution flow-field, along with other information related to the problem such as the use of flow-vector lengths to determine aspect ratio and size.

Explicitly-referenced metrics like the latter are examples of metrics which incorporate solution information in their evaluation of quality. Although not connected explicitly to finite element error bounds, the metrics have the capability of incorporating information available to the application, including solution properties. Since the metrics are based on the target matrix, they can measure the shape, size, and orientation of an element with respect to these same properties as embedded within the target.

Mesh quality is also important in the formulation of mesh quality improvement algorithms. For example, if one uses a tetrahedral shape measure, one can detect so-called sliver elements within a tetrahedral mesh. When a sliver element is found, a common improvement technique is to apply various local element swaps such as the 2-3, 3-2, or 4-4 operation. After the swap is complete, the shape metric is re-computed and, if the shape has improved, the swap is retained. Otherwise one reverts back to the initial sliver. Quality metrics can also be useful for mesh optimization methods that involve only mesh vertex movement. The attraction of such techniques is that the objective function that is optimized can be based on local quality metrics that are combined into a global measure of mesh quality. Not all quality metrics are suitable for incorporation into an objective function since many of them lead to non-convex optimization problems.

7. Conclusions

A brief overview of mesh *quality* has been given in this work. Quality metrics that are based purely on geometric characteristics are in widespread use in industry for production simulations. Use of these metrics is valid for automatic defect detection, generation or improvement of meshes based on expert judgment, and on non-adaptive calculations where the solution is slowly-varying and nearly-isotropic. Use of solution-independent metrics for problems in which 'quality' strongly depends on solution characteristic is invalid and should be avoided. Even where the application of a priori mesh quality metrics is valid, there are a number of technical issues which make their use somewhat unsatisfying. These include redundancy of metrics, poor formulations and definitions of some metrics, the widespread use of implicitly referenced metrics, and the lack of justification of threshold values that inform us whether or not to reject an element.

The primary role of a mesh is to enable an accurate simulation to be performed on a computer. As such, it is appropriate to consider mesh quality in terms of error analyses that have been performed for finite difference, finite volume, and finite element methods. The analysis for finite difference truncation error indicates no strong connection with a prior quality metrics that have been developed for curvilinear coordinates. The situation in finite elements is slightly better in that some upper bounds on the error do correspond to known mesh quality metrics, particularly the Ciarlet bounds that contain the explicitly-referenced condition number metric and the norm of the matrix B. However, the connection between other mesh quality metrics and the finite element error bounds is less clear. It is likely that for many metrics, relation to the bounds is impossible.

Recently, explicitly-referenced mesh quality metrics based on the matrix B have been introduced. These metrics use a target-matrix as the means of explicit referencing. The target matrices can be constructed based on information known to the specific application, including solution characteristics. The relationship of most of these metrics to finite element or other error analyses is unknown, yet, because they can be based on solution characteristics, they must presently be considered an intermediate case between metrics that rely solely on geometric information and explicitly-referenced metrics such as condition number that appear in the finite element upper bounds to the interpolation error.

8. Future Work

On the theoretical side, the development of explicitly referenced metrics such as those appearing in the Target-matrix paradigm can be carried forward in several ways. First, the author and colleagues have already begun using metrics whose explicit reference is based on the solution Hessian matrix for r-adaptive meshing. These are derived from the bounds of Formaggia & Perotto.[9] Second, much work remains to be done on the subject of automatic target-matrix construction to enable full use of the information that is available to applications. Third, we wonder if it is possible to connect some of the alternative target-metrics to the finite element interpolation error theory? If the use of these targets and metrics can be shown to decrease the error, then there must be some connection.

On the pragmatic side, the author suggests that further work be done on justifying the threshold values that are used in a priori uses of mesh quality metrics. It would also be valuable to provide training towards ending the use of redundant and/or flawed mesh quality metrics and to ensure the valid use of such metrics.

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